An algebraic analysis of conchoids to algebraic curves

J. R. Sendra · J. Sendra

Abstract We study the conchoid to an algebraic affine plane curve $\mathcal{C}$ from the perspective of algebraic geometry, analyzing their main algebraic properties. Beside $\mathcal{C}$, the notion of conchoid involves a point A in the affine plane (the focus) and a non-zero field element d (the distance). We introduce the formal definition of conchoid by means of incidence diagrams. We prove that the conchoid is a 1-dimensional algebraic set having at most two irreducible components. Moreover, with the exception of circles centered at the focus A and taking d as its radius, all components of the corresponding conchoid have dimension 1. In addition, we introduce the notions of special and simple components of a conchoid. Furthermore we state that, with the exception of lines passing through A, the conchoid always has at least one simple component and that, for almost every distance, all the components of the conchoid are simple. We state that, in the reducible case, simple conchoid components are birationally equivalent to $\mathcal{C}$, and we show how special components can be used to decide whether a given algebraic curve is the conchoid of another curve.

1 Introduction

The notion of conchoid is classical and is derived from a fixed point, a plane curve, and a length in the following way. Let $\mathcal{C}$ be a plane curve (the base curve), A a fixed point

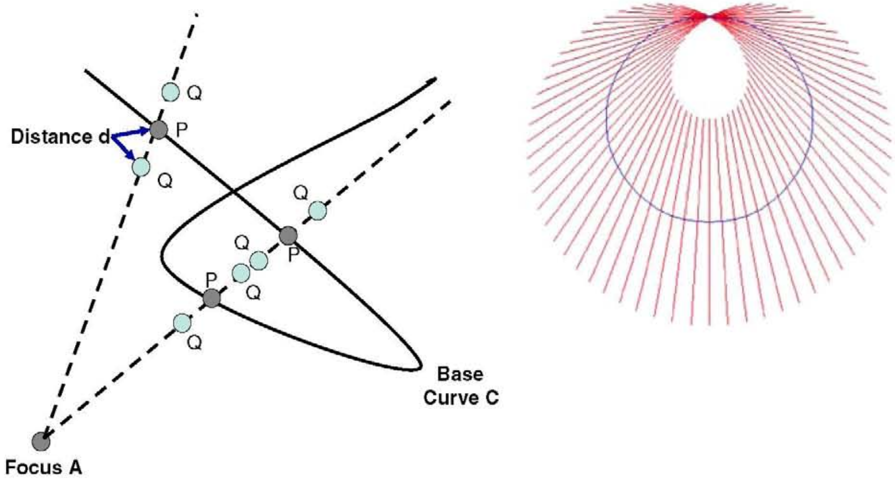


Fig. 1 *Left* conchoid geometric construction. *Right* conchoid of a circle with focus on it

in the plane (the focus), and d a non-zero fixed field element (the distance). Then, the conchoid of $\mathcal{C}$ from the focus A at distance d is the (closure of) set of points Q in the line AP at distance d of a point P varying in the curve $\mathcal{C}$ (see Fig. 1).

Classically, conchoids are introduced for curves in the real affine plane, while in this paper the curves are considered in the affine plane over an algebraically closed field $\mathbb{K}$ of characteristic zero. Throughout the paper, curves are considered reduced; that is, they are the zero set in $\mathbb{K}^2$ of non-constant square-free polynomials of $\mathbb{K}[y_1, y_2]$. Furthermore, for the definition of conchoid, we will assume that the base curve is irreducible, i.e., defined by an irreducible polynomial over $\mathbb{K}$. Also, if $\mathcal{C}$ is defined by the square-free polynomial $f(y_1, y_2)$, when we speak about the components of $\mathcal{C}$, we mean the curves defined by the non-constant irreducible factors (over $\mathbb{K}$) of $f(y_1, y_2)$

The two classical and most famous conchoids are the *Conchoid of Nicomedes* (see Example 2 and Fig. 3) and the *Limaçon of Pascal* (see Example 3, Fig. 1 right, Fig. 4) which appear when the base curve $\mathcal{C}$ is a line and a circle, respectively. Conchoid of Nicomedes was introduced by Nicomedes, around 200 B.C., to solve the problems of doubling the cube and of trisecting an angle. Conchoids play an important role in many applications as construction of buildings (one can already find specific methods for producing Limaçons in Albert Dürer's *Underweysung der Messung*), astronomy , electromagnetic research physics , optics , engineering in medicine and biology , mechanical in fluid processing . etc.

Although conchoids have been extensively used and applied in different areas, a deep theoretical analysis of the concept and its main properties is missing; at least from our point of view. In this paper, we consider conchoids from the perspective of algebraic geometry, and we study their main algebraic properties, with the aim of building a solid bridge from theory to practice that can be used for further theoretical and applied developments. More precisely, we introduce the formal definition of conchoid of an irreducible algebraic affine plane curve, over an algebraically closed field of characteristic zero, by means of incidence diagrams. We also introduce the notion of generic conchoid, and we show how elimination theory techniques, as Gröbner bases, can be applied to compute conchoids. We prove that, with the exception of a circle centered at the focus A and taking d as its radius, the conchoid is an algebraic curve having at most two irreducible components. Note that for the particular circle, mentioned above, the conchoid consists of two components: a circle of radius $2d$ and the zero-dimensional set consisting of A. In addition, we introduce the notions of special and simple components of a conchoid. Essentially, a component of a conchoid is special if its points are generated by more than one point of the original curve. This phenomenon appears when one computes conchoids of conchoids. Moreover we state that, with the exception of lines passing through the focus, the conchoid always has at least one simple component. Furthermore we prove that, for almost every distance and with the exception of lines passing through the focus, all the components of the conchoid are simple. Simple and special components play an important role in the study of conchoids. On one hand, simple components are related to the birationality of the maps in the incidence diagram (for instance, if a conchoid has two components, its simple components are birationally equivalent to the initial curve) and, on the other hand, special components can be used to decide whether a given algebraic curve is the conchoid of another curve.

The notions of simple and special components are inspired in the case of offset curves Furthermore, many of the properties presented here follow a similar pattern to those proved . However, as it is shown in Lemma 1, the two geometric constructions (conchoid and offset) only coincides when the base curve is a circle centered at the focus. In the case of offsets, the establishment of these properties has allowed us to provide formulas for the genus and for developing parametrization algorithm . We plan to investigate this, for the case of conchoids, in our future research. For the reader not being familiar with offsetting processes, we recall here the intuitive idea of an offset. Given $\mathcal{C}$ as above and $d \in \mathbb{K}^*$, the offset to $\mathcal{C}$ at distance d is the (closure of) set of points Q in the normal line to $\mathcal{C}$ at P, and at distance d of P, where P is a point varying in the curve $\mathcal{C}$.

The paper is structured as follows. In Sect. 2, we formally introduce the notion of conchoid of an algebraic curve. In Sect. 3 we state the basic algebraic properties of conchoids. In Sect. 4, we introduce the notions of simple and special components and we state their main properties. Section 5 studies how simple components are related to the birationality of the maps in the incidence diagram, and Sect. 6 shows how special components can be applied to detect whether a curve is the conchoid of another curve.

2 Definition of conchoid

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. In $\mathbb{K}^2$ we consider the symmetric bilinear form $B((x_1, x_2), (y_1, y_2)) = x_1y_1 + x_2y_2$, which induces a metric vector space with light cone $\mathfrak{L}$ of isotropy

$$\mathfrak{L} = \{(x_1, x_2) \in \mathbb{K}^2 \mid x_1^2 + x_2^2 = 0\}.$$

In this context, the circle of center $(a_1, a_2) \in \mathbb{K}^2$ and radius $d \in \mathbb{K}$ is the plane curve defined by $(x_1 - a_1)^2 + (x_2 - a_2)^2 = d^2$. We will say that the distance between the points $\bar{x}, \bar{y} \in \mathbb{K}^2$ is $d \in \mathbb{K}$ if $\bar{y}$ is on the circle of center $\bar{x}$ and radius d. Notice that the "distance" is hence defined up to multiplication by ± 1. On the other hand, if $\bar{x} \in \mathbb{K}^2$ is not isotropic (i.e. $\bar{x} \notin \mathfrak{L}$) we denote by $\|\bar{x}\|$ any of the elements in $\mathbb{K}$ such that $\|\bar{x}\|^2 = B(\bar{x}, \bar{x})$, and if $\bar{x} \in \mathbb{K}^2$ is isotropic (i.e. $\bar{x} \in \mathfrak{L}$), then $\|\bar{x}\| = 0$. In this paper we usually work with both solutions of $\|\bar{x}\|^2 = B(\bar{x}, \bar{x})$. For this reason we use the notation $\pm\|\bar{x}\|$.

In this situation, let $\mathcal{C}$ be the affine irreducible plane curve defined by the irreducible polynomial $f(\bar{y}) \in \mathbb{K}[\bar{y}]$, $\bar{y} = (y_1, y_2)$, let $d \in \mathbb{K}^*$ be a non-zero field element, and let $A = (a, b) \in \mathbb{K}^2$. In order to get a formal definition of the conchoid, one introduces the following incidence diagram:

$$\begin{tikzcd}
 & \mathfrak{B}(\mathcal{C}) \subset \mathbb{K}^2 \times \mathbb{K}^2 \times \mathbb{K} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\pi_1(\mathfrak{B}(\mathcal{C})) \subset \mathbb{K}^2 & & \mathcal{C} \subset \mathbb{K}^2
\end{tikzcd}$$

(Incidence Diagram)

where the conchoid incidence variety is

$$\mathfrak{B}(\mathcal{C}) = \left\{ (\bar{x}, \bar{y}, \lambda) \in \mathbb{K}^2 \times \mathbb{K}^2 \times \mathbb{K} \;\middle/\; \begin{array}{l} f(\bar{y}) = 0 \\ \|\bar{x} - \bar{y}\|^2 = d^2 \\ \bar{x} = A + \lambda(\bar{y} - A) \end{array} \right\}$$

and

$$\begin{array}{llll} \pi_1 : \mathbb{K}^2 \times \mathbb{K}^2 \times \mathbb{K} \longrightarrow \mathbb{K}^2, & & \pi_2 : \mathbb{K}^2 \times \mathbb{K}^2 \times \mathbb{K} \longrightarrow \mathbb{K}^2 & \\ \quad (\bar{x}, \bar{y}, \lambda) \longmapsto \bar{x} & & \quad (\bar{x}, \bar{y}, \lambda) \longmapsto \bar{y}. & \end{array}$$

The first equation in $\mathfrak{B}(\mathcal{C})$ corresponds to $\mathcal{C}$, the second guarantees that the distance between $\bar{x}$ and $\bar{y}$ is d, and the third expresses parametrically that $\bar{x}, \bar{y}, A$ are collinear. Then, we introduce the conchoid as follows.

Definition 1 *Let $\mathcal{C}$ be an affine irreducible plane curve, $d \in \mathbb{K}^*$, and $A \in \mathbb{K}^2$. We define the* conchoid of $\mathcal{C}$ from the focus A and distance d *as the algebraic Zariski closure in $\mathbb{K}^2$ of $\pi_1(\mathfrak{B}(\mathcal{C}))$, and we denote it by $\mathfrak{C}(\mathcal{C})$; i.e.*

$$\mathfrak{C}(\mathcal{C}) = \overline{\pi_1(\mathfrak{B}(\mathcal{C}))}.$$

Remark 1 Observe that

1. **[General assumption of the focus and the distance].** When A and d are not considered as parameters, and are just precise elements in $\mathbb{K}^2$ and $\mathbb{K}^*$, respectively, they can assumed to be, in a suitable system of affine coordinates, $A = (0, 0)$ and $d = 1$. So, since this will almost always be the case, we will take w.l.o.g. A as the origin O, and d as 1. Whenever we will not use this assumption we will explicitly say so. Indeed, if d is generic, the conchoid will be denoted by $\mathfrak{C}(\mathcal{C}, d)$ (see Sect. 4), and when both, A and d, are generic, the conchoid will be denoted by $\mathfrak{C}(\mathcal{C}, A, d)$ (see Sect. 6).
2. **[General assumption on the isotropic lines through the focus].** $\mathfrak{C}(\mathcal{C}) = \emptyset$ iff $\mathfrak{B}(\mathcal{C}) = \emptyset$ iff $\mathcal{C}$ is one of two lines $\mathcal{L}^{\pm}$ given by $(y_1 - a) \pm \sqrt{-1}(y_2 - b) = 0$. So, in the sequel, we assume that $\mathcal{C} \neq \mathcal{L}^{+}$ and $\mathcal{C} \neq \mathcal{L}^{-}$.
3. **[Extension of the definition].** In Definition 1 we have considered irreducible curves. The same reasoning can be done for reducible curves, introducing the conchoid as the union of the conchoids of the irreducible components.
4. **[Computation of the Conchoid].** Let I be the ideal in $\mathbb{K}[\bar{x}, \bar{y}, \lambda]$ generated by the polynomials defining $\mathfrak{B}(\mathcal{C})$. Then, by the Closure Theorem one has that $\mathfrak{C}(\mathcal{C}) = V(I \cap \mathbb{K}[\bar{x}])$. Hence elimination theory techniques, such as Gröbner bases, provide the conchoid. Also, one may consider the equation of the line in $\mathfrak{B}(\mathcal{C})$ given implicitly in order to have a system with one less variable (namely λ). In this case, one has to take into account that, when $A \in \mathcal{C}$, the circle $\|\bar{x} - A\|^2 = d^2$ appears as a fake component of $\mathfrak{C}(\mathcal{C})$, and thus it has to be crossed out.
5. **[Generic Conchoid].** one may introduce the notion of generic conchoid. Let us consider d as a new variable. Now, $\mathfrak{B}(\mathcal{C})$ is seen as an algebraic set in $\mathbb{K}^2 \times \mathbb{K}^2 \times \mathbb{K} \times \mathbb{K}$; we denote it by $\mathfrak{B}(\mathcal{C})_G$. Then, the generic conchoid is defined as

$$\mathfrak{C}(\mathcal{C})_G = \overline{\pi_1(\mathfrak{B}(\mathcal{C})_G)}.$$

   Now, if I_G is the ideal in $\mathbb{K}[\bar{x}, \bar{y}, \lambda, d]$ generated by the polynomials defining $\mathfrak{B}(\mathcal{C})_G$, by the Closure Theorem (see [4, p. 122]), one has that $\mathfrak{C}(\mathcal{C})_G = V(I_G \cap \mathbb{K}[\bar{x}, d])$.
   , one gets that for almost all values of $d \in \mathbb{K}^*$ the generic conchoid specializes properly (see Example 1). An example where the specialization behaves improperly is when $\mathcal{C}$ is taken as a circle centered at A and d its radius (compare to Theorem 1). A similar reasoning might be done with a generic focus, nevertheless we do not consider this situation here.

Let us illustrate the definition with two examples.

Example 1 Let $\mathcal{C}$ be the parabola over $\mathbb{C}$ defined by $f(y_1, y_2) = y_2 - y_1^2$, let $A = (0, -1)$ and $d = 1/2$. Then $\mathfrak{B}(\mathcal{C})$ is defined by

$$f(\bar{y}) = 0, \ (x_1 - y_1)^2 + (x_2 - y_2)^2 = \frac{1}{4}, \quad x_1 = \lambda y_1, x_2 = -1 + \lambda(y_2 + 1).$$

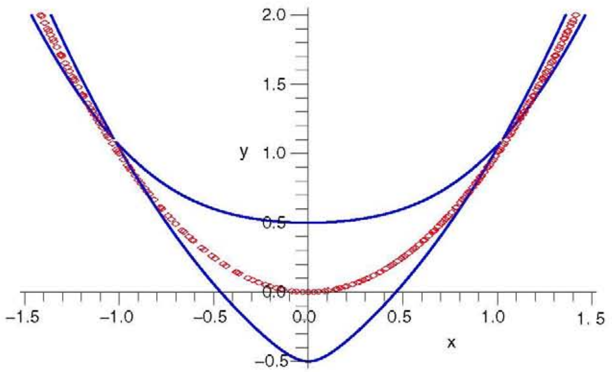


Fig. 2 $y_2 = y_1^2$ (in *dots*) and $\mathfrak{C}(C)$ with $A = (0, -1)$, and $d = 1/2$ (in *continuous traced*)

Now, considering $\lambda > y_1 > y_2 > x_1 > x_2$, and computing a Gröbner basis w.r.t. the lex order, one gets that $\mathfrak{C}(C)$ is defined by (see Fig. 2):

$$\begin{aligned}g(x_1, x_2) = {} & 16x_1^8 + 32x_2^2x_1^6 + 16x_1^4x_2^4 + 32x_2x_1^6 - 32x_1^2x_2^5 + 24x_1^6 - 24x_1^4x_2^2\\ & - 96x_1^2x_2^4 + 16x_2^6 - 8x_2x_1^4 - 120x_1^2x_2^3 + 64x_2^5 + 25x_1^4 - 68x_1^2x_2^2\\ & + 92x_2^4 + 48x_2^3 + 12x_1^2 - 8x_2^2 - 16x_2 - 4,\end{aligned}$$

which is an irreducible curve over $\mathbb{C}$. Similarly, one gets that $\mathfrak{C}(C)_G$ is given by

$$\begin{aligned}g_G(x_1, x_2, d) = {} & -d^2 + x_2^2 + 4\,x_2^3 - 2\,x_2\,x_1^2 - 4\,x_2\,d^2 - 6\,x_2^2x_1^2 - 6\,x_2^2d^2 + 6\,x_2^4\\ & + 8\,x_1^2x_2\,d^2 - 8\,x_1^2x_2^3 + 3\,x_1^2d^2 + x_1^4 + 4\,x_2^5 - x_2^2x_1^4 - 6\,x_2^4x_1^2\\ & + 7\,x_2^2x_1^2d^2 - 2\,x_1^4x_2\,d^2 + 2\,x_1^6 + 2\,x_1^6x_2 + 2\,x_1^4d^2 - 2\,x_1^2x_2^5\\ & + 2\,x_1^6x_2^2 + x_1^4x_2^4 - 2\,x_2^2d^2x_1^4 + 2\,x_2^3d^2x_1^2 + x_2^6 - x_2^4d^2\\ & - 2\,x_1^6d^2 + x_1^8 + x_1^4d^4 - 4\,x_2^3d^2.\end{aligned}$$

Note that $g_G(\bar{x}, 1/2) = g(\bar{x})$.

In the following example, we take C as a line such that $A \notin C$, obtaining the well known *Conchoid of Nicomedes*.

Example 2 (Conchoid of Nicomedes) Let C be the line defined by $f(y_1, y_2) = y_2$, $A = (0, 1)$ and $d = 2$. Then, $\mathfrak{C}(C)$ is defined by (see Fig. 3):

$$g(x_1, x_2) = x_2^2x_1^2 + x_2^4 - 2x_2^3 - 3x_2^2 + 8x_2 - 4.$$

We finish the section studying the connection of conchoids to offsets (see [1] for further details on offsets). We represent the offset to C at distance d as $\mathcal{O}_d(C)$.

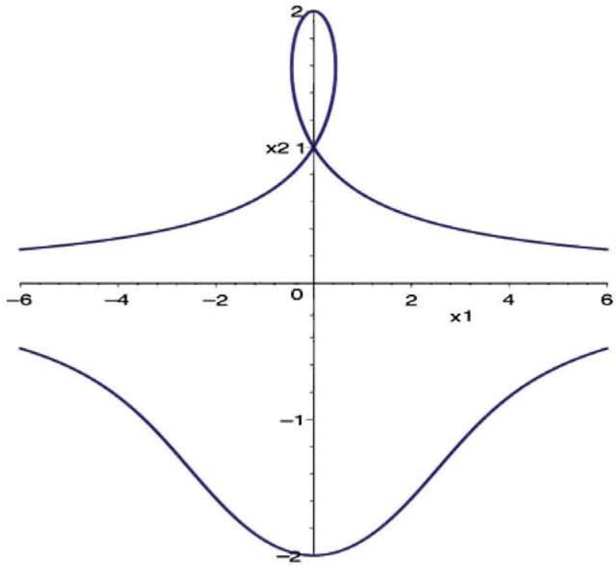


Fig. 3 Conchoid of the line $y_2 = 0$ with $A = (0, 1)$ and $d = 2$

Lemma 1 *Let $\mathcal{C}$ be a real irreducible curve, and $d \in \mathbb{R}^*$. Then, $\mathfrak{C}(\mathcal{C}) = \mathcal{O}_d(\mathcal{C})$ if and only if $\mathcal{C}$ is a circle centered at the focus. Hence, in this case, the 1-dimensional components of $\mathfrak{C}(\mathcal{C})$ are also circles.*

Proof If $\mathcal{C}$ is a circle centered at the focus the result is obvious. Reciprocally, if $\mathfrak{C}(\mathcal{C}) = \mathcal{O}_d(\mathcal{C})$,

3 Basic properties of conchoids

In this section we state the first basic properties on conchoids. For this purpose, let $\mathcal{C}$ be an irreducible affine plane curve and $f(\bar{y})$ its defining polynomial. Moreover, with the exception of Example 3, we assume w.l.o.g. that the focus A is the origin O and $d = 1$ (see Remark 1 (1)), and that $\mathcal{C}$ is none of the isotropy lines $\mathcal{L}^+$, $\mathcal{L}^-$ (see Remark 1 (2)). We start analyzing the projections appearing in the incidence diagram (see Sect. 2).

Lemma 2 *Let π_1, π_2 be the projections in the incidence diagram.*

(1) *π_1 is, at most $(2:1)$, over all points in $\pi_1(\mathfrak{B}(\mathcal{C}))\setminus\{O\}$.*
(2) *If $O \in \pi_1(\mathfrak{B}(\mathcal{C}))$, and $\mathcal{C}$ is not the unit circle centered at O, then $\mathrm{Card}(\pi_1^{-1}(O)) \leq 2\deg(\mathcal{C})$.*
(3) *π_2 is $(2:1)$ over all points in $\mathcal{C}\setminus\{\mathcal{L}^+ \cup \mathcal{L}^-\}$.*

Proof (1) Let $\bar{x}^0 \in \pi_1(\mathfrak{B}(\mathcal{C}))\setminus\{O\}$. Then, $\pi_1^{-1}(\bar{x}^0)$ is included in the intersection of the circle $\|\bar{y} - \bar{x}^0\|^2 = 1$ with the line $\bar{x}^0 = \lambda\bar{y}$ (note that $\bar{x}^0 \neq O$ and hence it is a line). Thus, by Bézout's theorem, $\mathrm{Card}(\pi_1^{-1}(\bar{x}^0)) \leq 2$.

(2) Let $O \in \pi_1(\mathfrak{B}(\mathcal{C}))$. Then, $\pi_1^{-1}(O) \subset \{(O, \bar{y}, 0) \mid f(\bar{y}) = 0, \|\bar{y}\|^2 = 1\}$. So, since $\mathcal{C}$ is irreducible and it is not the circle $\|\bar{y}\|^2 = 1$, by Bézout's theorem, $\mathrm{Card}(\pi_1^{-1}(O)) \leq 2\deg(\mathcal{C})$.

(3) Let $\bar{y}^0 \in \mathcal{C}\backslash\{\mathcal{L}^{\pm}\}$. So $\|\bar{y}^0\| \neq 0$ and $\pi_2^{-1}(\bar{y}^0) = \{(\lambda\bar{y}^0, \bar{y}^0, \lambda) \mid \lambda = 1 \pm \frac{1}{\|\bar{y}^0\|}\}$.

Remark 2 Note that, by Lemma 2 (3), $\pi_2(\mathfrak{B}(\mathcal{C})) = \mathcal{C}\backslash\{\mathcal{L}^+ \cup \mathcal{L}^-\}$.

The following theorem essentially states that the conchoid is a curve with at most two irreducible components.

Theorem 1 (1) *All the components of $\mathfrak{B}(\mathcal{C})$ have dimension* 1.

(2) *If $\mathcal{C}$ is not the unit circle centered at O, all the components of $\mathfrak{C}(\mathcal{C})$ have dimension* 1.

(3) *If $\mathcal{C}$ is the unit circle centered at O, $\mathfrak{C}(\mathcal{C})$ decomposes as the union of $\{O\}$ and the circle centered at* 0 *and radius* 2.

(4) *$\mathfrak{C}(\mathcal{C})$ has at most two components.*

Proof (1) By assumption $\mathcal{C} \neq \mathcal{L}^{\pm}$. So $\mathfrak{B}(\mathcal{C}) \neq \emptyset$. Let Γ be an irreducible component of $\mathfrak{B}(\mathcal{C})$. By Lemma 2 (3), $\dim(\Gamma) \leq \dim(\mathcal{C}) = 1$. Now let $M \in \Gamma$, $P = \pi_2(M)$ and $\mathcal{P}(t) = (P_1(t), P_2(t))$ a place of $\mathcal{C}$ centered at P. Let $\Delta(t) = P_1(t)^2 + P_2(t)^2$, and

$$\mathcal{Q}^{\pm}(t) = \left(\left(1 \pm \frac{1}{\sqrt{\Delta}}\right)\mathcal{P}(t), \mathcal{P}(t), 1 \pm \frac{1}{\sqrt{\Delta}}\right).$$

By Remark 2, $\|P\| \neq 0$. So $\mathcal{Q}^{\pm}(t)$ parametrizes locally two curves contained in $\mathfrak{B}(\mathcal{C})$ and passing through each of the two points (see Lemma 2 (3)) in $\pi_2^{-1}(P)$, respectively. Thus, $\dim \Gamma \geq 1$. So, $\dim(\Gamma) = 1$.

(2) follows as (1), using that π_1 is finite (see Lemma 2).

(3) It is trivial.

(4) The reasoning is analogous to Theorem 1 in [10], using Lemma 2.

Next lemma follows from Lemma 2, Theorem 1, and the theorem on the dimension of fibres

Lemma 3 *Let π_1, π_2 be the projections in the incidence diagram, and let Ω, Ω^* be non-empty open subsets of $\mathcal{C}$ and of a 1-dimensional irreducible component of $\mathfrak{C}(\mathcal{C})$, respectively. Then,*

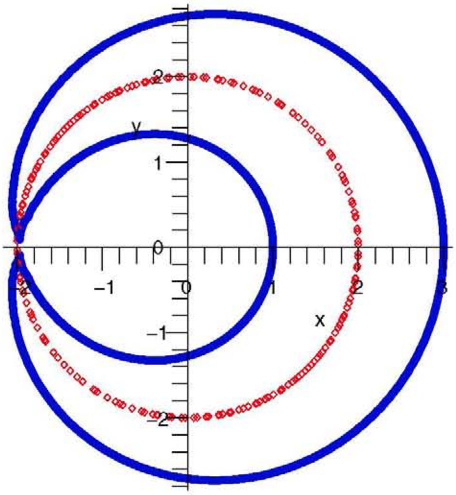
(1) *$\pi_1(\pi_2^{-1}(\Omega))$ is a non-empty Zariski dense subset of $\mathfrak{C}(\mathcal{C})$.*

(2) *$\pi_2(\pi_1^{-1}(\Omega^*))$ is a non-empty Zariski dense subset of $\mathcal{C}$.*

Proof (1) $\overline{\pi_2^{-1}(\Omega)} = \mathfrak{B}(\mathcal{C})$. So $\pi_1(\pi_2^{-1}(\Omega))$ is constructible in $\mathfrak{C}(\mathcal{C})$.

(2) Let $\mathcal{M}$ be a 1-dimensional irreducible component of $\mathfrak{C}(\mathcal{C})$, and $\Omega' = \Omega^* \cap \pi_1(\mathfrak{B}(\mathcal{C})))$. Since $\dim(\mathcal{M}) = 1$ and π_1 is finite (see Lemma 2), there exists an irreducible component $\Gamma \subset \pi_1^{-1}(\Omega')$ of dimension 1. Since π_2 is finite (see Lemma 2), and $\dim(\mathcal{C}) = 1$, then $\dim(\pi_2(\Gamma)) = 1$. Thus, (2) follows from $\pi_2(\Gamma) \subset \pi_2(\pi_1^{-1}(\Omega')) \subset \pi_2(\pi_1^{-1}(\Omega^*)) \subset \mathcal{C}$ and using that $\mathcal{C}$ is irreducible.

Fig. 4 Circle centered at $(0, 0)$ and radius $r = 2$ (in *dots*), and its conchoid from the focus $(-2, 0)$ and radius $d = 1$ (*continuous traced*)



Example 3 (Limaçons of Pascal) Let $\mathcal{C}$ be the circle centered at $(0, 0)$ and radius $r = 2$. Then, the conchoid of $\mathcal{C}$ with $A = (-2, 0) \in \mathcal{C}$ and $d = 1$ (*Limaçon of Pascal* at distance 1, see Fig. 4), is defined by the polynomial:

$$g(x_1, x_2) = x_1^4 + 2x_2^2x_1^2 - 9x_1^2 - 4x_1 - 9x_2^2 + 12 + x_2^4.$$

On the other hand, if we move the focus to $A = (0, 0) \notin \mathcal{C}$, one observes that the conchoid at distance $d = 1$ has two irreducible components (two circles centered at A and radius 1 and 3, respectively) defined by the irreducible factors of $(x_1^2 - 9 + x_2^2) \cdot (x_1^2 - 1 + x_2^2)$. Note that, in this case, $\mathfrak{C}(\mathcal{C})$ is the offset of $\mathcal{C}$ at distance $d = 1$ (see Lemma 1).

4 Simple and special components

We use the same notation and assumptions as in Sect. 3: $\mathcal{C}$ is irreducible, defined by $f(\bar{y})$, and $\mathcal{C} \neq \mathcal{L}^+$, $\mathcal{C} \neq \mathcal{L}^-$. As before, we take w.l.o.g. $A = O$ and $d = 1$. Also we assume w.l.o.g. that $\mathcal{C}$ is not the unit circle centered at O (see Theorem 1). However, in Lemma 4 and Theorem 4, d will be generic. So, in these statements the conchoid will be denoted by $\mathfrak{C}(\mathcal{C}, d)$, instead of $\mathfrak{C}(\mathcal{C})$, in order to emphasize this fact.

In this section, we introduce and analyze the notion of simple and special components of a conchoid. Special and simple components provide information on the birationality of the projections in the incidence diagram (see Sect. 5), and they can be used to decide whether a curve is a conchoid (see Sect. 6). Essentially, one component of the conchoid is special if its points are generated for more than one point of the original curve. This phenomenon appears when one computes conchoids of conchoids (see Theorem 2). In addition, Theorem 4 states that, for almost every distance and with the exception of lines passing through the focus, all the components of the conchoid are simple.

Definition 2 *An irreducible component* $\mathcal{M}$ *of* $\mathfrak{C}(\mathcal{C})$ *is called* simple *if there exists a non-empty Zariski dense subset* $\Omega \subset \mathcal{M}$ *such that, for* $Q \in \Omega$, $\mathrm{Card}(\pi_2(\pi_1^{-1}(Q))) = 1$. *Otherwise* $\mathcal{M}$ *is called* special.

Remark 3 $\mathcal{M}$ is special iff there exists a Zariski dense $\emptyset \neq \Omega \subset \mathcal{M}$ such that for $Q \in \Omega$, $\mathrm{Card}(\pi_2(\pi_1^{-1}(Q))) > 1$. Moreover, since $\mathcal{M} \setminus (\mathcal{L}^+ \cup \mathcal{L}^-) \neq \emptyset$, the above Ω can be taken as $\mathcal{M} \setminus (\mathcal{L}^+ \cup \mathcal{L}^-)$.

Proposition 1 *If* $\mathcal{M}$ *is an irreducible component of* $\mathfrak{C}(\mathcal{C})$, *then* $\mathcal{C} \subset \mathfrak{C}(\mathcal{M})$.

Proof Let $\mathcal{M}_0 := \mathcal{M} \setminus \mathcal{L}^{\pm}$, and π_i, π_i^* be the projections in the incidence diagram of $\mathcal{C}$ and $\mathcal{M}$, respectively. By Lemma 3 (2), $\mathcal{C}_1 := \pi_2(\pi_1^{-1}(\mathcal{M}_0))$ is a non-empty Zariski dense in $\mathcal{C}$; observe that $\mathcal{C}$ is not the circle $\|\bar{y}\|^2 = 1$, so $\dim(\mathcal{M}) = 1$. Now, for $P \in \mathcal{C}_1$, there exist $Q_0 \in \mathcal{M}_0$ and $\lambda_0 \in \mathbb{K}$ such that $(Q_0, P, \lambda_0) \in \mathfrak{B}(\mathcal{C})$. Since, $Q \notin \mathcal{L}^{\pm}$ then $\lambda_0 \neq 0$. So $(P, Q_0, \lambda_0^{-1}) \in \mathfrak{B}(\mathcal{M})$. Thus, $\mathcal{C}_1 \subset \pi_1^*(\mathfrak{B}(\mathcal{M}))$, and taking closures $\mathcal{C} \subset \mathfrak{C}(\mathcal{M})$. □

Remark 4 Note that, by Proposition 1 and Lemma 1, if $\mathcal{C}$ is not a circle centered at the focus, then for every $d \in \mathbb{K}^*$ none component of the conchoid of $\mathcal{C}$ is a circle centered at the focus.

Next theorem shows that, similarly as in the offsetting construction special components appear only when computing conchoids of conchoids.

Theorem 2 *An irreducible component* $\mathcal{M}$ *of* $\mathfrak{C}(\mathcal{C})$ *is special iff* $\mathfrak{C}(\mathcal{M}) = \mathcal{C}$.

Proof Let $\mathcal{M}$ be special and Ω as in Remark 3. By Proposition 1, $\mathcal{C} \subset \mathfrak{C}(\mathcal{M})$. To see that $\mathfrak{C}(\mathcal{M}) \subset \mathcal{C}$, let π_i and π_i^* the projections in the incidence diagram of $\mathcal{C}$ and $\mathcal{M}$, respectively. Let $\Omega^* = \pi_1^*(\pi_2^{*-1}(\Omega)) \subset \mathfrak{C}(\mathcal{M})$, which is dense by Lemma 3. For $P \in \Omega^* \setminus \{O\}$, there exists $Q \in \Omega$ (in particular $Q \neq O$) and $\lambda_0 \in \mathbb{K}$ such that $(P, Q, \lambda_0) \in \mathfrak{B}(\mathcal{M})$. Let $\mathcal{D}$ be the circle $\|\bar{y} - Q\|^2 = 1$. Now, since $Q \in \Omega$, there exist $P_1, P_2 \in \mathcal{C}$, $P_1 \neq P_2$, and $\lambda_1, \lambda_2 \in \mathbb{K}$ such that $(Q, P_i, \lambda_i) \in \mathfrak{B}(\mathcal{C})$. So Q, P_1, P_2, P, O are collinear and $P, P_1, P_2 \in \mathcal{D}$. Then, $P \in \{P_1, P_2\} \subset \mathcal{C}$. Thus, $\Omega^* \setminus \{O\} \subset \mathcal{C}$, and taking closures $\mathfrak{C}(\mathcal{M}) \subset \mathcal{C}$.

Conversely, let $\Omega = \mathcal{M} \setminus (\mathcal{L}^{\pm} \cup \{P \,/\, \|P\|^2 = 1\})$. By Remark 4, $\Omega \neq \emptyset$ and dense in $\mathcal{M}$. Let $P \in \Omega$, then $P \neq O$. By Lemma 2 (3), there exist $Q_1, Q_2 \in \mathfrak{C}(\mathcal{M})$, $Q_1 \neq Q_2$, and $\lambda_1, \lambda_2 \in \mathbb{K}$, with $(Q_i, P, \lambda_i) \in \mathfrak{B}(\mathcal{M})$. Since $\|P\|^2 \neq 1$ then $\lambda_i \neq 0$. Thus, since $\mathfrak{C}(\mathcal{M}) = \mathcal{C}$, $(P, Q_i, \lambda_i^{-1}) \in \mathfrak{B}(\mathcal{C})$, and hence $\mathrm{Card}(\pi_2(\pi_1^{-1}(P)) > 1$. □

We illustrate the previous results by an example.

Example 4 Let $\mathcal{C}$ be the unit circle centered at O. First, we compute $\mathfrak{C}(\mathcal{C})$, with focus at $A = (-1, 0)$ and distance $d = 2$ obtaining a Limaçon of Pascal, defined by the polynomial:

$$g(x_1, x_2) = x_1^4 + 2\,x_2^2\,x_1^2 - 6\,x_1^2 - 8\,x_1 - 6\,x_2^2 - 3 + x_2^4.$$

Note that we get a cardioid; let us denote it as $\mathcal{C}'$ (see Fig. 5 left).

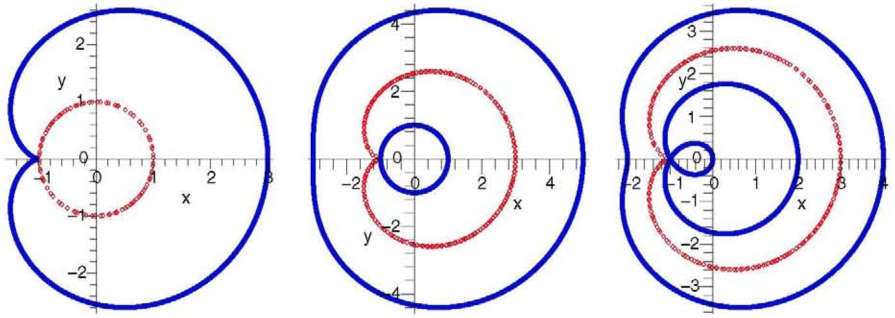


Fig. 5 *Left* circle centered at $(0, 0)$ and radius $r = 1$ (in *dots*), and its conchoid from the focus $(-1, 0)$ and radius $d = 2$ (cardioid). *Center* cardioid (in *dots*), and its conchoid from the focus $(-1, 0)$ and radius $d = 2$ (*continuous traced*). *Right* cardioid (in *dots*), and its conchoid from the focus $(-1, 0)$ and radius $d = 1$ (*continuous traced*)

Now, we compute the conchoid of the cardioid C' from the same focus $A = (-1, 0)$ and the same distance $d = 2$. In this case, one gets a reducible curve, say C'', with two irreducible components defined by the irreducible factors of:

$$(x_1^2 + x_2^2 - 1)(x_1^4 + 2\,x_2^2\,x_1^2 - 18\,x_1^2 - 32\,x_1 - 18\,x_2^2 - 15 + x_2^4).$$

Note that, one component of C'' is the initial circle C (and therefore it is a special component of C''; see Theorem 2), and the other component is a Limaçon of Pascal of C from the focus A and distance $d = 4$. See Fig. 5 center. On the other hand, the conchoid of C', from the focus A but now taking distance $d = 1$, decomposes as the union of two irreducible components defined by the irreducible factors of the equation:

$$(x_1^4 + 2\,x_2^2\,x_1^2 - 11\,x_1^2 - 18\,x_1 - 11\,x_2^2 - 8 + x_2^4)$$
$$(x_1^4 + 2\,x_2^2\,x_1^2 - 3\,x_1^2 - 2\,x_1 - 3\,x_2^2 + x_2^4),$$

which correspond to two Limaçons of Pascal of C from the focus A at distance $d = 3$ and $d = 1$ respectively; see Fig. 5 right.

Next theorem states the main property of the components of a conchoid.

Theorem 3 *If C is not a line through the focus, then $\mathfrak{C}(C)$ has at least one simple component.*

Proof Since C is not a line through O, we may assume that C meets the y_1-axis $\mathcal{L}$ in a point $P = (c, 0)$ with $c \neq 0$. Since $C \cap \mathcal{L}$ is finite and $\mathbb{K}$ has characteristic zero, the set $B = \{n \in \mathbb{Z} \;/\; (c + n, 0) \in C\}$ is finite. Let $n_0 = \max(B)$, then $Q = (c + n_0 + 1, 0) \in \mathfrak{C}(C)$. Furthermore, the component of $\mathfrak{C}(C)$ passing through Q is simple, since otherwise $(c + n_0 + 2, 0) \in C$ that contradicts the definition of n_0. □

Corollary 1 *If C is not a line through the focus, and $\mathfrak{C}(C)$ is irreducible, then $\mathfrak{C}(C)$ is simple.*

Remark 5 If $\mathcal{C}$ is a line through the focus, then $\mathfrak{C}(\mathcal{C})$ is irreducible and special. So $\mathfrak{C}(\mathcal{C}) = \mathcal{C}$.

Corollary 2 *The only curves whose conchoids are irreducible and special are the lines through the focus.*

In the following two results, d is treated generically. So, we use the notation $\mathfrak{C}(\mathcal{C}, d)$, instead of $\mathfrak{C}(\mathcal{C})$, to emphasize this fact.

Lemma 4 *If $\mathcal{C}$ is not a line through the focus, there exist at most finitely many $d \in \mathbb{K}^*$ for which all the conchoids $\mathfrak{C}(\mathcal{C}, d)$ have a common component.*

Proof Let $D \subset \mathbb{K}^*$ be an infinite set, and $\mathcal{M}$ an irreducible component of $\mathfrak{C}(\mathcal{C}, d)$ for every $d \in D$. Let $d_1, \ldots, d_{r+1} \in D$ such that $d_i^2 \neq d_j^2,\ \forall i \neq j$, where $r = \deg(\mathcal{C})$. Let $\Omega = \bigcap_{i=1}^{r+1} \pi_{1,i}(\pi_{2,i}^{-1}(\mathcal{C})) \cap (\mathcal{M} \setminus \{O\})$, where $\pi_{1,i}, \pi_{2,i}$ are the projections of the incidence diagram of $\mathfrak{C}(\mathcal{C}, d_i)$. Since $\mathcal{M}$ is irreducible, by Lemma 3, Ω is a non-empty Zariski dense of $\mathcal{M}$. Now, let $Q \in \Omega$, and $\mathcal{L}$ the line passing through Q and O; note that $Q \neq O$. Because of the definition of incidence variety, for every $i \in \{1, \ldots, r+1\}$ there exists $P_i \in \mathcal{C}$ such that $P_i \in \mathcal{L}$ and $\|Q - P_i\|^2 = d_i^2$. Moreover, since $d_i^2 \neq d_j^2$, then $P_i \neq P_j$ if $i \neq j$. Thus, $\mathrm{Card}(\mathcal{L} \cap \mathcal{C}) > \deg(\mathcal{C})$ and hence $\mathcal{C} = \mathcal{L}$, in contradiction with the hypothesis.

Theorem 4 *If $\mathcal{C}$ is not a line passing through the focus, for almost every distance $d \in \mathbb{K}^*$ all the components of $\mathfrak{C}(\mathcal{C}, d)$ are simple.*

Proof Let $D \subset \mathbb{K}^*$ be an infinite set such that, for $d \in D$, $\mathfrak{C}(\mathcal{C}, d)$ has a special component. Let $r = \deg(\mathcal{C})$, $\delta := 1 + \binom{r}{2}$, and $d_1, \ldots, d_\delta \in D$ such that: $d_i^2 \neq d_j^2$, $\mathcal{C}$ is not a component of $\mathfrak{C}(\mathcal{C}, d_i)$, and $\mathcal{M}_i \neq \mathcal{M}_j$ being $\mathcal{M}_i$ the special component of $\mathfrak{C}(\mathcal{C}, d_i)$; this is always possible because of Lemma 4. Let $\Omega_{i,0} = \mathcal{M}_i \setminus \{\mathcal{L}^{\pm}\}$, as in Remark 3, let $\mathcal{C}_0 = \mathcal{C} \setminus \{\mathcal{L}^{\pm}\}$, and let

$$\Delta_i = (\mathcal{M}_i \setminus \Omega_{i,0}) \cup (\mathcal{M}_i \cap \mathcal{C}) \bigcup_{j \neq i} (\mathcal{M}_i \cap \mathcal{M}_j).$$

Since $\mathcal{M}_i, \mathcal{C}$ are irreducible, $\mathcal{M}_i \neq \mathcal{M}_j$, $\mathcal{M}_i \neq \mathcal{C}$, $\mathcal{C} \neq \mathcal{L}^{\pm}$, and $\mathcal{M}_i \neq \mathcal{L}^{\pm}$, $\Delta := \bigcup_{i=1}^{\delta} \Delta_i \cup (\mathcal{C} \setminus \mathcal{C}_0)$ is a finite set. We take a line $\mathcal{L}$ passing through O and such that $(\mathcal{L} \setminus \{O\}) \cap \Delta = \emptyset$. Let $Q_i \in \mathcal{L} \cap \Omega_{i,0}$ for $i \in \{1, \ldots, \delta\}$; in particular, $Q_i \neq O$. By construction $\mathrm{Card}(\{Q_1, \ldots, Q_\delta\}) = \delta$. Since $\mathcal{M}_i$ is special, there exist $P_i', P_i \in \mathcal{C}$, $P_i' \neq P_i$, such that $\|Q_i - P_i'\|^2 = \|Q_i - P_i\|^2 = d_i^2$ and $Q_i, P_i, P_i' \in \mathcal{L}$. Let C_i be the circle centered at Q_i and radius d_i. So, we have δ different circles with all the centers at $\mathcal{L}$. This implies that $\mathrm{Card}(\mathcal{L} \bigcap_{i=1}^{\delta} C_i) \geq r + 1$. Moreover $\mathcal{L} \cap C_i = \{P_i, P_i'\} \subset \mathcal{C}$. Thus $\mathrm{Card}(\mathcal{L} \cap \mathcal{C}) \geq r + 1$, and hence $\mathcal{C} = \mathcal{L}$, in contradiction with the hypothesis.

5 The role of simple components

Simple components play an important role, from the theoretical point of view, in the study of conchoids. Essentially, they provide information on the birationality of the maps in the incidence diagram, and hence they open the door for studying, in further research, algebraic and geometric properties of the conchoids.

Lemma 5 *Let π_1, π_2 be the projections in the incidence diagram of $\mathcal{C}$, and $\mathcal{M}$ an irreducible component of $\mathfrak{C}(\mathcal{C})$.*

(1) *If $\mathfrak{C}(\mathcal{C})$ is reducible, the restricted map $\pi_2|_{\pi_1^{-1}(\mathcal{M})} : \pi_1^{-1}(\mathcal{M}) \longrightarrow \mathcal{C}$ is birational.*
(2) *The restricted map $\pi_1|_{\pi_1^{-1}(\mathcal{M})} : \pi_1^{-1}(\mathcal{M}) \longrightarrow \mathcal{M}$ is birational iff $\mathcal{M}$ is simple.*

Proof (1) follows using Theorem 1 (1), that ensures that all components of $\mathfrak{B}(\mathcal{C})$ have dimension 1, and by Lemma 2 (3), that ensures that π_2 is generically (2:1).
(2) follows from the notions of simple component and birational map.

From this lemma, one directly deduces the following corollary.

Corollary 3 *Let $\mathcal{C}$ be such that $\mathfrak{C}(\mathcal{C})$ is reducible. Then, the simple components of $\mathfrak{C}(\mathcal{C})$ are birationally equivalent to $\mathcal{C}$.*

In the following example we illustrate these results.

Example 5 Let $\mathcal{C}$ be the plane curve defined by

$$f(y_1, y_2) = -3 + 9\, {y_1}^2 + 9\, {y_2}^2 + 2\, y_2 - 4\, {y_2}^4 - 4\, {y_1}^4 - 8\, {y_1}^2 {y_2}^2.$$

Let $A = (0, -1)$ and $d = 1/2$. The conchoid of $\mathcal{C}$ from A at distance d decomposes as $\mathcal{M} \cup \mathcal{N}$ where $\mathcal{N}$ is defined by $N(\bar{x}) := x_1^2 + x_2^2 - 1$ and $\mathcal{M}$ by $M(\bar{x}) := {x_1}^4 + 2\, {x_2}^2 {x_1}^2 - 3\, {x_1}^2 - 3\, {x_2}^2 - 2\, x_2 + {x_2}^4$. $\mathcal{N}$ is special and $\mathcal{M}$ is simple. The incidence variety decomposes as $\Gamma_1 \cup \Gamma_2$ where $\Gamma_1 := \overline{\pi_1^{-1}(\mathcal{N})}$ and $\Gamma_2 := \overline{\pi_1^{-1}(\mathcal{M})}$. By Lemma 5 (2), $\pi_1|_{\Gamma_2} : \Gamma_2 \longrightarrow \mathcal{M}$ is birational and $\pi_1|_{\Gamma_2} : \Gamma_1 \longrightarrow \mathcal{N}$ is not. Indeed, we can find a rational inverse

$$\begin{aligned}(\pi_1|_{\Gamma_2})^{-1} : \mathcal{M} &\longrightarrow \Gamma_2 \\ \bar{x} &\longmapsto \left(\bar{x}, -\frac{x_1\left({x_1}^2 - 6 + {x_2}^2 - 4\, x_2\right)}{4(x_2+1)}, -\frac{{x_2}^2 - 4x_2 + {x_1}^2 - 2}{4}, \right. \\ &\qquad \left. \frac{-4x_2 - 4}{x_1^2 - 6 + x_2^2 - 4x_2}\right)\end{aligned}$$

Thus, by Corollary 3,

$$\begin{aligned}\varphi : \mathcal{M} &\longrightarrow \mathcal{C} \\ \bar{x} &\longmapsto \left(-\frac{x_1\left({x_1}^2 - 6 + {x_2}^2 - 4\, x_2\right)}{4(x_2+1)}, -\frac{{x_2}^2 - 4x_2 + {x_1}^2 - 2}{4}\right)\end{aligned}$$

is birational (i.e. $\varphi = \pi_2|_{\Gamma_2} \circ (\pi_1|_{\Gamma_2})^{-1}$). In fact, we can find a rational inverse

$$\begin{aligned}\varphi^{-1} : \mathcal{C} &\longrightarrow \mathcal{M}\\ \bar{y} &\longmapsto \left(\frac{y_1\,(3+8\,y_2+4\,y_1{}^2+4\,y_2{}^2)}{8(y_2+1)},\, y_2 - \frac{5}{8} + \frac{1}{2}\,y_1{}^2 + \frac{1}{2}\,y_2{}^2 \right).\end{aligned}$$

6 Detecting conchoids

In this section, we show how special components can be used to decide whether a given irreducible plane curve $\mathcal{D}$ is the conchoid of another curve. So, both the focus A and the distance d are treated generically, and hence we will denote the conchoid by $\mathfrak{C}(\mathcal{C}, A, d)$, instead of $\mathfrak{C}(\mathcal{C})$, in order to emphasize this fact.

First observe that one can always find a curve $\mathcal{C}$, and $A \in \mathbb{K}^2$, $d \in \mathbb{K}^*$ such that $\mathcal{D}$ is a component of $\mathfrak{C}(\mathcal{C}, A, d)$. For this purpose, one simply has to take $\mathcal{C}$ as an irreducible component of $\mathfrak{C}(\mathcal{D}, A, d)$. Then, by Proposition 1, $\mathcal{D} \subset \mathfrak{C}(\mathcal{C}, A, d)$. So, we are now interested in deciding whether there exist $A \in \mathbb{K}^2$, $d \in \mathbb{K}^*$ and $\mathcal{C}$ such that $\mathcal{D} = \mathfrak{C}(\mathcal{C}, A, d)$. By Theorem 2, this is equivalent to decide whether there exist $A \in \mathbb{K}^2$, $d \in \mathbb{K}^*$ such that $\mathfrak{C}(\mathcal{D}, A, d)$ has a special component; if so, $\mathcal{C}$ is the special component. We proceed as follows.

Finding the focus. Let $g(\bar{x})$ be the defining polynomial of $\mathcal{D}$, and $A = (a, b)$ where a, b are unknowns. We consider a line $\mathcal{L}$ passing through A and a generic point $Q = (z_1, z_2)$, expressed parametrically as $L(t) = A + t(Q - A)$. Now, we take two different points $P_1 := L(t_1)$, $P_2 := L(t_2)$ on $\mathcal{L}$, and we consider the algebraic set

$$\mathcal{S} = \left\{ (a, b, z_1, z_2, t_1, t_2, \omega) \in \mathbb{K}^7 \;\middle|\; \begin{array}{l} \mathrm{Eq}_1 := g(L(t_1)) = 0 \\ \mathrm{Eq}_2 := g(L(t_2)) = 0 \\ \mathrm{Eq}_3 := \|L(t_1) - Q\|^2 = \|L(t_2) - Q\|^2 \\ \mathrm{Eq}_4 := \omega \cdot (t_1 - t_2) = 1. \end{array} \right\}$$

Eq_1 and Eq_2 ensure that $P_1, P_2 \in \mathcal{D}$, Eq_3 requires that P_1, P_2 are on the same circle centered at Q, and Eq_4 guarantees that $P_1 \neq P_2$. Therefore, if $\pi : \mathbb{K}^7 \to \mathbb{K}^2$ is the projection $\pi(a, b, z_1, z_2, t_1, t_2, \omega) = (a, b)$, then the possible focuses A, such that there exists d for which $\mathfrak{C}(\mathcal{D}, A, d)$ has a special component, belong to $\overline{\pi(\mathcal{S})}$; i.e., belong to $V(I \cap \mathbb{K}[a, b])$, where I is the ideal generated by $\{\mathrm{Eq}_1, \ldots, \mathrm{Eq}_4\}$

Example 6 Let $\mathcal{D}$ be the line defined by $g(x_1, x_2) = \lambda\, x_1 + \mu\, x_2 + \rho$. Then,

$$\begin{aligned}\mathrm{Eq}_1 &= \lambda\; (a + t_1\,(z_1 - a)) + \mu\; (b + t_1\,(z_2 - b)) + \rho\\ \mathrm{Eq}_2 &= \lambda\; (a + t_2\,(z_1 - a)) + \mu\; (b + t_2\,(z_2 - b)) + \rho\\ \mathrm{Eq}_3 &= (a + t_1\,(z_1 - a) - z_1)^2 + (b + t_1\,(z_2 - b) - z_2)^2\\ &\quad - (a + t_2\,(z_1 - a) - z_1)^2 - (b + t_2\,(z_2 - b) - z_2)^2\\ \mathrm{Eq}_4 &= \omega\,(t_1 - t_2) - 1.\end{aligned}$$

Moreover,
$\{\mu\, b + \lambda\, a + \rho, \lambda\, z_1 + \mu\, z_2 + \rho, (-z_2 + b)^2\,(t_1 + t_2 - 2)\,, (-z_2 + b)^2\,\left(-2\,\omega + 1 + 2\,\omega t_2\right)\,, \omega t_1 - \omega t_2 - 1\}$
is a Gröbner basis of $\{\mathrm{Eq}_1, \ldots, \mathrm{Eq}_4\}$ w.r.t. lexorder, with $\omega > t_1 > t_2 > z_1 > z_2 > a > b$. Hence the possible focuses (a, b) satisfy $\mu\, b + \lambda\, a + \rho = 0$; i.e., they are on the line $\mathcal{D}$. Indeed all of them are valid (see Remark 5). □

Detecting conchoids. Let $\mathcal{D}$, $g(\bar{x})$, L, $\mathrm{Eq}_1, \ldots, \mathrm{Eq}_4$ be as above. We want to decide whether $\mathcal{D}$ is a conchoid from a given focus $A = (a, b)$. For this purpose, we decide whether there exists $d \in \mathbb{K}^*$ such that $\mathfrak{C}(\mathcal{D}, A, d)$ has a special component.

Let $L^*, \mathrm{Eq}_1^*, \ldots, \mathrm{Eq}_4^*$ be the polynomials obtained specializing at A the polynomials $L, \mathrm{Eq}_1, \ldots, \mathrm{Eq}_4$. Let $\mathcal{S}^* \subset \mathbb{K}^5$ be the algebraic set defined by $\{\mathrm{Eq}_1^*, \ldots, \mathrm{Eq}_4^*\}$, let $\pi^* := \mathbb{K}^5 \to \mathbb{K}^2$ where $\pi^*(z_1, z_2, t_1, t_2, \omega) = (z_1, z_2)$, and let I^* be the ideal generated by $\{\mathrm{Eq}_1^*, \ldots, \mathrm{Eq}_4^*\}$. Reasoning similarly as in the previous subsection, one deduces that the algebraic Zariski closure

$$\mathcal{H} := \overline{\pi^*(\mathcal{S}^*)} = V(I^* \cap \mathbb{K}[z_1, z_2])$$

contains the special components of $\mathfrak{C}(\mathcal{D}, A, d)$. Thus, for each irreducible component $\mathcal{H}$ we compute its generic conchoid (see Remark 1) to afterwards checking whether for some $d \in \mathbb{K}^*$ we get $\mathcal{D}$.

Note that $\mathcal{H}$ may also contain the curve defining the geometric locus of those points Q such that the intersection of $\mathcal{D}$ with the line passing though A, Q contains two points P_1, P_2 with $\|P_1 - Q\| = \|P_2 - Q\|$. On the other hand, we also know that the generic conchoid specializes properly for all values of d, but finitely many exceptions. These exceptions can be determined, but the computation may be too heavy.

Example 7 Let $\mathcal{D}$ be defined by $g(\bar{x}) = {x_1}^4 + 2\,{x_2}^2{x_1}^2 - 9\,{x_1}^2 - 4\,x_1 - 9\,{x_2}^2 + 12 + {x_2}^4$ and $A = (-2, 0)$. Computing a Gröbner basis of the ideal I^* generated by $\{\mathrm{Eq}_1^*, \ldots, \mathrm{Eq}_4^*\}$ w.r.t. lexorder, with $\omega > t_1 > t_2 > z_1 > z_2$, we get that $\mathcal{H} := V(I^* \cap \mathbb{K}[z_1, z_2])$ decomposes as the union of the circle $\mathcal{H}_1$ defined by $-4 + {z_1}^2 + {z_2}^2$ and the quartic $\mathcal{H}_2$ defined by $-4 - 4\,z_1 + 15\,{z_1}^2 + 16\,{z_1}^3 + 4\,{z_1}^4 - {z_2}^2 + 16\,{z_2}^2 z_1 + 8\,{z_2}^2{z_1}^2 + 4\,{z_2}^4$. The generic conchoid $\mathfrak{C}(\mathcal{H}_1, A, d)$ is given by

$$\begin{aligned} G(\bar{x}, d) = {} & -8\,{x_1}^2 - 8\,{x_2}^2 - 4\,d^2 + 16 - 4\,x_1 d^2 + {x_1}^4 \\ & + 2\,{x_2}^2{x_1}^2 - {x_1}^2 d^2 - {x_2}^2 d^2 + {x_2}^4. \end{aligned}$$

Solving the algebraic system in d provided by $g(\bar{x}) = G(\bar{x}, d)$ one gets that $d = \pm 1$. Indeed $\mathcal{D} = \mathfrak{C}(\mathcal{H}_1, A, 1)$. Performing the same computations with $\mathcal{H}_2$ one gets that $\mathcal{D}$ is not a conchoid of $\mathcal{H}_2$.

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